Regenerative compositions in the case of slow variation: A renewal theory approach

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Abstract

A regenerative composition structure is a sequence of ordered partitions derived from the range of a subordinator by a natural sampling procedure. In this paper, we extend previous studies [1, 8, 12] on the asymptotics of the number of blocks K_n in the composition of integer n, in the case when the Lévy measure of the subordinator has a property of slow variation at 0. Using tools from the renewal theory the limit laws for K_n are obtained in terms of integrals involving the Brownian motion or stable processes. In other words, the limit laws are either normal or other stable distributions, depending on the behavior of the tail of Lévy measure at ∞. Similar results are also derived for the number of singleton blocks.

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1 Introduction

Let $S := (S(t))_{t\geq 0}$ be a subordinator (i.e. an increasing Lévy process) with $S(0) = 0$, zero drift, no killing and a nonzero Lévy measure ν on $\mathbb{R}_+$. The closed range $\mathcal{R}$ of the process S is a regenerative subset of $\mathbb{R}_+$ of zero Lebesgue measure. The range $\mathcal{R}$ splits the positive halfline in infinitely many disjoint component intervals that form an open set $(0,\infty)\setminus\mathcal{R}$. These component intervals, further called gaps, are associated with jumps of S. Let $E_1,\dots,E_n$ be a sample drawn independently of S from the standard exponential distribution. Each sample point E_j falls in the generic gap (a, b) with probability $e^{-a}-e^{-b}$. A gap is said to be occupied if it contains at least one of n sample points. The sequence of positive occupancy numbers of the gaps, recorded in the natural order of the gaps, is a composition (ordered partition) $\mathcal{C}_n$ of integer n. The number K_n counting the blocks of the composition is equal to the number of gaps occupied by at least one sample point, and

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the number $K_{n,r}$ counting the blocks of size r is the number of gaps occupied by exactly r out of n sample points, so that $K_n = \sum_{r=1}^n K_{n,r}$ and $n = \sum_{r=1}^n rK_{n,r}$.

The sequence of random compositions $(\mathcal{C}_n)_{n\in\mathbb{N}}$ derived in this way has the following two recursive properties. The first property of *sampling consistency* is a form of exchangeability: removing a randomly chosen sample point from the first n points maps $\mathcal{C}_n$ in a distributional copy of $\mathcal{C}_{n-1}$. This property is obvious from the construction and exchangeability, because removing a random point has the same effect as restricting to $n-1$ points $E_1, \dots, E_{n-1}$. The second property is specific for regenerative $\mathcal{R}$. Consider composition $\mathcal{C}_n$ of size n and suppose it occurs that the first part, which is the number of sample points in the leftmost occupied gap, is some $m < n$, then deleting this part yields a composition on $n-m$ remaining points which is a distributional copy of $\mathcal{C}_{n-m}$. This property is a combinatorial counterpart of the regenerative property of $\mathcal{R}$, therefore sequences $(\mathcal{C}_n)_{n\in\mathbb{N}}$ are called *regenerative composition structures* [11]. In particular, for suitable choice of ν the construction generates an ordered version of the familiar Ewens-Pitman two-parameter partition structure [11].

The regenerative composition structures appear in a variety of contexts related to partition-valued processes and random discrete distributions (see [6] for a survey). To bring the construction of compositions in a more conventional context we may consider a random distribution function $F(t) = 1 - \exp(-S(t))$ on positive reals, also known as a *neutral to the right prior* [17]. Since F has atoms, an independent n-sample from the distribution (defined conditionally given F) will have clusters of repeated values, thus we may define a composition $\mathcal{C}_n$ by recording the multiplicities in the order of increase of the values represented in the sample.

Limit distributions for K_n (properly centered and normalized) were studied under various assumptions on S. When the Lévy measure ν is finite, the process S is compound Poisson, and $\mathcal{R}$ is the discrete set of atoms of a renewal point process. A characteristic feature of this case is that almost all of the gaps within $[0, S(\log n)]$ are occupied, and these give a dominating contribution to K_n. In the compound Poisson case there is a rather complete theory [7, 8, 10] surveyed in [9].

In the case of infinite Lévy measure the asymptotic behaviour of K_n is related to that of the tail $\nu[x,\infty)$ as $x \to 0$; concrete results for infinite ν have been obtained under the assumption of regular variation. If $\nu[x,\infty)$ varies regularly at 0 with positive index, both K_n and $K_{n,1}$ may be normalized by the same constant (no centering required) to entail convergence to multiples of the same random variable, which may be represented as the exponential functional of a subordinator [13]. This case is relatively easy, because the number of occupied gaps within the partial range in $[S(t_1), S(t_2)]$ is of the same order of growth, as $n \to \infty$, for every time interval $0 \le t_1 < t_2 \le \infty$.

The case of infinite Lévy measure with $\nu[x,\infty)$ slowly varying at 0 is much more delicate, because the occupied gaps do not occur that uniformly as in the case of regular variation with positive index, nor the primitive gap-counting works: unlike the compound Poisson case, $\mathcal{R}$ has topology of a Cantor set. Each $\mathbb{E}K_{n,r}$ is then of the order of growth smaller than that of $\mathbb{E}K_n$, and the convergence of K_n and $K_{n,r}$'s requires nontrivial centering. Normal limits were shown in [12] in the special case of subordinators which, like the gamma subordinators, have $\nu[x,\infty)$ of logarithmic growth as $x \to 0$. Normal limits for K_n for wider families of slowly varying functions were obtained in [1] under the assumption

that the subordinator has finite variance and the Laplace exponent of S satisfies certain smoothness and growth conditions. It turned that the case of slow variation required a further division, with qualitatively different scaling functions in each subcase [1].

The method of [1] relied on linearization of the compensator process for the number of occupied gaps contained in $[0, S(t)]$, and application of the functional central limit theorem for S. In this paper we develop a different approach to the asymptotics of K_n in the case of $\nu[x, \infty)$ slowly varying at 0. As in [1], we analyse the compensator process, but instead of linearizing it, apply the renewal theory and functional limit theorems for the *first passage times* process, that is random function inverse to S. This gives a big technical advantage, enabling us to simplify arguments and to increase generality. The class of slowly varying functions covered in this paper will be larger than that in [1, 12]. In particular, we will omit the assumption of finite variance of S and find conditions on the Lévy measure ν to guarantee a weak convergence of K_n to the normal or some other stable distributions. A similar approach, with a discrete-time version of the compensator will be applied also in the case of finite Lévy measure, leading to known asymptotics [7, 8, 10] in a more compact way. We shall also identify the limit distribution for $K_{n,1}$ in terms of an integral involving a random process corresponding to the limit law of K_n. With some additional effort, our approach to the limit laws of $K_{n,1}$ could be extended to $K_{n,r}$ for all $r \geq 1$, but to avoid technical complications we do not pursue this extension here, as our main focus is the development of the new method.

2 Preliminaries

As in much of the previous work, it will be convenient to poissonize the occupancy model, that is to replace the exponential sample $E_1, \ldots, E_n$ of fixed size n by atoms of an inhomogeneous Poisson process $(\pi_t(x))_{x\geq 0}$, which is independent of S and has the intensity measure $\lambda_t(\mathrm{d}x) = te^{-x}\mathrm{d}x$ on $\mathbb{R}_+$. The total number of atoms, $\pi_t := \pi_t(\infty)$, has then the Poisson distribution with mean t. We will use the notation $K(t) := K_{\pi_t}$ for the number of gaps occupied by at least one atom of the Poisson process, and $K(t, r)$ for the number of gaps occupied by exactly r such atoms.

Introduce

$$\Phi(t) := \int_{[0,\infty)} \big(1 - \exp\{-t(1 - e^{-x})\}\big)\nu(\mathrm{d}x), \quad t > 0.$$

By Proposition 2.1 of [1] the increasing process $(A(t, u))_{u\in[0,\infty]}$ defined by

$$A(t, u) := \int_{[0,\,u]} \Phi\big(te^{-S(v)}\big)\mathrm{d}v$$

is the compensator of the increasing process which counts the number of gaps in $[0, S(u)] \setminus \mathcal{R}$, that are occupied by at least one atom of the Poisson sample. Similarly, one can check that

$$A^{(r)}(t, u) := \int_{[0,\,u]} \Phi^{(r)}\big(te^{-S(v)}\big)\frac{(te^{-S(v)})^r}{r!}\mathrm{d}v,$$

where $\Phi^{(r)}$ denotes the rth derivative of Φ, is the compensator of the increasing process which counts the number of gaps in $[0, S(u)] \setminus \mathcal{R}$ that contain exactly r Poisson atoms.

The asymptotics of $K(t)$ and $K(t,1)$ for large t is closely related to the terminal values of the compensators

$$A(t) := A(t,\infty) = \int_{[0,\infty)} \Phi(te^{-S(v)})\mathrm{d}v = \int_{[0,\infty)} \Phi(te^{-s})\,\mathrm{d}T(s) \tag{1}$$

and

$$A^{(1)}(t) := A^{(1)}(t,\infty) = \int_{[0,\infty)} \Phi'(te^{-S(v)})te^{-S(v)}\,\mathrm{d}v, \tag{2}$$

where

$$T(s) := \inf\{t \geq 0 : S(t) > s\}, \quad s \geq 0$$

is the passage time of S through level s.

Like in many other models of allocating 'balls' in 'boxes' with random probabilities of 'boxes', the variability of K_n has two sources: the randomness of $\mathcal{R}$, and the randomness involved in drawing a sample conditionally given $\mathcal{R}$. For regenerative compositions it has been shown, in various forms, that the first factor of variability has a dominant role. See, for instance, [8] for the compound Poisson case. We shall confirm the phenomenon in the case of slow variation by showing that $A(n)$ absorbs a dominant part of the variability, to the extent that $A(n)$ and K_n, normalized and centered by the same constants, have the same limiting distributions.

Throughout we shall assume that the function Φ satisfies one of the following three conditions:

CONDITION A:

$$\varphi(t) := \Phi(e^t) \ \sim \ t^\beta L_1(t), \quad t \to \infty,$$

for some $\beta \in [0,\infty)$ and some function L_1 slowly varying at ∞. For $\beta = 0$ we assume $\lim_{t\to\infty} L_1(t) = \infty$.

CONDITION B: $\varphi(t)$ belongs to de Haan's class Γ, i.e., there exists a measurable function $h : \mathbb{R} \to (0,\infty)$ called the *auxiliary function* of φ such that

$$\lim_{t\to\infty} \frac{\varphi(t-uh(t))}{\varphi(t)} = e^{-u} \ \text{ for all } u \in \mathbb{R} \quad \text{and} \quad \lim_{t\to\infty} h(t) = \infty.$$

CONDITION C: $\varphi(t)$ is a bounded function, which holds if and only if the Lévy measure ν is finite, *i.e.* S is a compound Poisson process.

Let

$$\widehat{\Phi}(t) := \int_0^\infty (1-e^{-tx})\nu(\mathrm{d}x)$$

denote the conventional Laplace exponent of S. Then

$$\Phi(t) \sim \widehat{\Phi}(t), \quad t \to \infty.$$

Therefore, conditions A, B and C can be equivalently formulated with $\widehat{\Phi}$ in place of Φ. Either of the conditions implies that Φ is a function slowly varying at ∞, hence by Karamata's Tauberian theorem

$$\nu[x,\infty] \ \sim \ \Phi(1/x), \quad x \to 0. \tag{3}$$

It should be noted that there are slowly varying functions which satisfy neither of the conditions A, B and C, for instance functions which behave like

$$\Phi(t) \sim \exp\left(\int_2^t \frac{|\sin u|}{\log u}\mathrm{d}u\right), \quad t \to \infty.$$

However, functions that are not covered by one of the conditions A, B and C are rather exceptional. The case $\beta = 0$ of Condition A covers the functions called in [1] 'slowly growing', and the case $0 < \beta < \infty$ 'moderately growing'.

The rest of the paper is organized as follows. In Section 3 we consider the case of infinite ν when one of the conditions A or B holds, and derive the limit distributions for K_n and $K_{n,1}$. In Section 4 we give a simplified treatment of the much studied case [7, 10, 8] when ν is a finite measure, that is Condition C holds. Finally, some auxiliary facts are collected in the Appendix.

3 Subordinators with infinite Lévy measure

3.1 Convergence of K_n

Our first main result concerns 'slowly growing' or 'moderately growing' functions Φ of slow variation, behaving like e.g. $\Phi(t) \sim (\log_k t)^\beta (\log_m t)^\delta$, for some $\beta > 0$, $\delta \geq 0$, $k \in \mathbb{N}$, $m \in \mathbb{N}_0$, where $\log_i x$ denotes the i-fold iteration of the natural logarithm.

Introduce the moments of $S(1)$

$$\mathtt{s}^2 := \operatorname{Var} S(1) = \int_{[0,\infty)} x^2 \nu(\mathrm{d}x), \quad \mathtt{m} := \mathbb{E} S(1) = \int_{[0,\infty)} x\nu(\mathrm{d}x).$$

Note that $\mathtt{m} < \infty$ under the assumptions of all the subsequent theorems.

Theorem 3.1. *Suppose Condition A holds.*
(a) *Suppose* $\mathtt{s}^2 < \infty$. *If* $\beta > 0$ *then*

$$\frac{K_n - \mathtt{m}^{-1}\int_{[1,n]} y^{-1}\Phi(y)\mathrm{d}y}{\sqrt{\mathtt{s}^2\mathtt{m}^{-3}\log n \Phi(n)}} \overset{d}{\to} \beta\int_{[0,1]} Z(1-y)y^{\beta-1}\mathrm{d}y, \quad n \to \infty,$$

where $\big(Z(y)\big)_{y\in[0,1]}$ *is the Brownian motion, and if* $\beta = 0$ *then the limiting random variable is* $Z(1)$.
(b) *Suppose* $\mathtt{s}^2 = \infty$ *and*

$$\int_0^x y^2\nu(\mathrm{d}y) \ \sim \ L(x), \quad x \to \infty,$$

for some L *slowly varying at* ∞. *Let* $c(x)$ *be any positive function satisfying* $\lim_{x\to\infty} xL(c(x))/c^2(x) = 1$. *If* $\beta > 0$ *then*

$$\frac{K_n - \mathtt{m}^{-1}\int_{[1,n]} y^{-1}\Phi(y)\mathrm{d}y}{\mathtt{m}^{-3/2}c(\log n)\Phi(n)} \overset{d}{\to} \beta\int_{[0,1]} Z(1-y)y^{\beta-1}\mathrm{d}y, \quad n \to \infty,$$

where $(Z(y))_{y\in[0,1]}$ is the Brownian motion, and if $\beta=0$ then the limiting random variable is $Z(1)$.
(c) *Suppose*

$$\nu[x,\infty] \ \sim\ x^{-\alpha}L(x), \quad x\to\infty, \tag{4}$$

for some L slowly varying at ∞ and $\alpha\in(1,2)$. Let $c(x)$ be any positive function satisfying $\lim_{x\to\infty} xL(c(x))/c^{\alpha}(x)=1$. If $\beta>0$ then

$$\frac{K_n-\mathtt{m}^{-1}\int_{[1,\,n]}y^{-1}\Phi(y)\mathrm{d}y}{\mathtt{m}^{-(\alpha+1)/\alpha}c(\log n)\Phi(n)} \ \overset{d}{\to}\ \beta\int_{[0,1]}Z(1-y)y^{\beta-1}\mathrm{d}y, \quad n\to\infty,$$

where $\big(Z(y)\big)_{y\in[0,1]}$ is the α-stable Lévy process such that $Z(1)$ has characteristic function

$$u\mapsto \exp\{-|u|^{\alpha}\Gamma(1-\alpha)(\cos(\pi\alpha/2)+i\sin(\pi\alpha/2)\operatorname{sgn}(u))\},\ u\in\mathbb{R}, \tag{5}$$

and if $\beta=0$ then the limiting random variable is $Z(1)$.

Remark 3.2. Set $J:=\beta\int_{[0,1]}Z(1-y)y^{\beta-1}\mathrm{d}y$. By Lemma 5.1,

$$\log\mathbb{E}\exp(\mathrm{i}tJ)=\int_{[0,\,1]}\log\mathbb{E}\exp(\mathrm{i}t(1-x)^{\beta}Z(1))\mathrm{d}x.$$

Hence $J\overset{d}{=}(\alpha\beta+1)^{-1/\alpha}Z(1)$ where the case $\alpha=2$ corresponds to parts (a) and (b) of Theorem 3.1.

In the definition of constants Φ can be replaced by the Laplace exponent $\widehat{\Phi}$, since the difference between the functions vanishes at ∞ (see [13] Lemma A.1 or [1] Lemma 2.3).

Proof. Under the assumptions of part (a) we denote by $Z(\cdot)$ the Brownian motion and set $g(t):=\sqrt{\mathtt{s}^2\mathtt{m}^{-3}t}$, under the assumptions of part (b) we denote by $Z(\cdot)$ the Brownian motion and set $g(t):=\mathtt{m}^{-3/2}c(t)$ and under the assumptions of part (c) we denote by $Z(\cdot)$ the α-stable Lévy process such that $Z(1)$ has characteristic function (5), and set $g(t):=\mathtt{m}^{-1-1/\alpha}c(t)$.

For later use we note that g varies regularly at ∞ with index $1/\alpha$, where $\alpha=2$ corresponds to the cases (a) and (b). This follows from Theorem 1.5.12 in [4] which is a result on asymptotic inverses of regularly varying functions. Further we note that in the cases (b) and (c) $g(x)$ grows faster than $\sqrt{x}$. In the latter case, this follows trivially from the regular variation of g with index $1/\alpha$, $\alpha\in(1,2)$. In the former case, we have $\int_{[0,\,x]}y^2\nu(\mathrm{d}y)\ \sim\ L(x)$, where $\lim_{x\to\infty}L(x)=\infty$, and $c(x)$ satisfies $\lim_{x\to\infty}\frac{xL(c(x))}{c^2(x)}=1$. Since $\lim_{x\to\infty}L(c(x))=\infty$ we infer $\lim_{x\to\infty}\frac{x}{c^2(x)}=0$.
Step 1: We first investigate convergence in distribution of, properly normalized and centered, $A(t)$, as $t\to\infty$. Recalling the notation $\varphi(t)=\Phi(e^t)$, representation (1) can be rewritten using integration by parts for the Lebesgue-Stieltjes integral as follows

$$A(e^t)-\varphi(0)T(t)=\int_{[0,\,t]}T(t-z)\mathrm{d}\varphi(z)+\int_{[t,\,\infty)}\varphi(t-z)\mathrm{d}T(z)=:A_1(t)+A_2(t).$$

Now we want to look at the asymptotic behavior of $A_2(t)$, as $t \to \infty$. Since $\Phi'(0) < \infty$ (this is equivalent to the characteristic property $\int_{[0,\infty)} \min(y,1)\nu(\mathrm{d}y) < \infty$ which holds for every Lévy measure ν), the function $\varphi(t)$ is integrable on $(-\infty, 0]$, which together with its monotonicity ensures that it is directly Riemann integrable on $(-\infty, 0]$. Therefore, by the key renewal theorem

$$\mathbb{E}A_2(t) = \mathbb{E}\int_{[t,\,\infty)} \varphi(t-z)\mathrm{d}T(z) \ \to\ \mathtt{m}^{-1}\int_{(-\infty,\,0]} \varphi(z)\mathrm{d}z < \infty, \quad t \to \infty. \tag{6}$$

Case $\beta > 0$. It is known (see Theorem 2a in [3]) that

$$W_t(\cdot) := \frac{T(t\cdot) - \mathtt{m}^{-1}(t\cdot)}{g(t)} \Rightarrow\ Z(\cdot), \quad t \to \infty, \tag{7}$$

in $D[0,\infty)$ in the Skorohod M_1-topology. In particular,

$$\frac{T(t) - \mathtt{m}^{-1}t}{g(t)\varphi(t)} \ \stackrel{P}{\to}\ 0, \quad t \to \infty. \tag{8}$$

To apply Lemma 5.3 take $X_t = W_t$ and let Y_t and Y be random variables with distribution functions $\mathbb{P}\{Y_t \le y\} = \frac{\varphi(ty)}{\varphi(t)} =: u_t(y)$ and $\mathbb{P}\{Y \le y\} = y^\beta =: u(y)$, $0 \le y \le 1$. Then, as $t \to \infty$,

$$\begin{aligned}\frac{A_1(t) - \mathtt{m}^{-1}\int_{[0,\,t]}(t-z)\mathrm{d}\varphi(z)}{g(t)\varphi(t)} &= \int_{[0,1]} W_t(1-y)\mathrm{d}u_t(y)\\ &\stackrel{d}{\to} \int_{[0,1]} Z(1-y)\mathrm{d}u(y)\\ &= \beta\int_{[0,1]} Z(1-y)y^{\beta-1}\mathrm{d}y = J.\end{aligned}$$

Recalling (6) and (8) we obtain

$$\frac{A(e^t) - \mathtt{m}^{-1}\Big(\int_{[0,\,t]}(t-z)\mathrm{d}\varphi(z) + \varphi(0)t\Big)}{g(t)\varphi(t)} \ \stackrel{d}{\to}\ J. \tag{9}$$

Noting that

$$\int_{[1,\,e^t]} y^{-1}\Phi(y)\mathrm{d}y = \int_{[0,\,t]} \varphi(y)\mathrm{d}y = \int_{[0,\,t]} (t-z)\mathrm{d}\varphi(z) + \varphi(0)t$$

and replacing in (9) e^t by t concludes the proof of Step 1 in the case $\beta > 0$.
Case $\beta = 0$. We have, for $\varepsilon \in (0,1)$

$$\begin{aligned}\frac{A_1(t) - \mathtt{m}^{-1}\int_{[0,\,t]}(t-z)\mathrm{d}\varphi(z)}{g(t)\varphi(t)} &= \frac{\int_{[0,\,\varepsilon]} W_t(1-y)\mathrm{d}\varphi(ty)}{\varphi(t)}\\ &+ \frac{\int_{[\varepsilon,\,1]} W_t(1-y)\mathrm{d}\varphi(ty)}{\varphi(t)}\\ &=: J_1(t,\varepsilon) + J_2(t,\varepsilon).\end{aligned} \tag{10}$$

We first show that

$$\lim_{\varepsilon\downarrow 0}\lim_{t\to\infty} J_1(t,\varepsilon) = Z(1) \ \text{ in distribution.} \tag{11}$$

To this end, we use the bounds

$$\inf_{y\in[0,\varepsilon]} W_t(1-y)\frac{\varphi(\varepsilon t)-\varphi(0)}{\varphi(t)} \le J_1(t,\varepsilon) \le \sup_{y\in[0,\varepsilon]} W_t(1-y)\frac{\varphi(\varepsilon t)}{\varphi(t)}.$$

Recall that the function $h_1 : D[0,\infty)\to\mathbb{R}$ defined by $h_1(x) := \sup_{y\in[0,\varepsilon]} x(y)$ is M_1-continuous (see Section 13.4 in [19]). Hence, in view of (7) we conclude that, as $t\to\infty$, the right-hand side converges in distribution to $\sup_{y\in[0,\varepsilon]} Z(1-y)$. This further converges to $Z(1)$ on letting $\varepsilon\downarrow 0$. A similar argument applies to the left-hand side, and (11) has been proved.

Using the inequality

$$\inf_{y\in[\varepsilon,1]} W_t(1-y)\frac{\varphi(t)-\varphi(\varepsilon t)}{\varphi(t)} \le J_2(t,\varepsilon) \le \sup_{y\in[\varepsilon,1]} W_t(1-y)\frac{\varphi(t)-\varphi(\varepsilon t)}{\varphi(t)}$$

and arguing in much the same way as above we conclude that $\lim_{t\to\infty} J_2(t,\varepsilon)=0$ in distribution. This together with (6) allows us to conclude that

$$\frac{A(e^t)-\mathtt{m}^{-1}\Big(\int_{[0,\,t]}(t-z)\mathrm{d}\varphi(z)+\varphi(0)t\Big)}{g(t)\varphi(t)} \ \stackrel{d}{\to}\ Z(1). \tag{12}$$

Replacing in this relation e^t by t completes the proof of Step 1 in the case $\beta=0$.
Step 2: Now we argue that the same convergence in distribution holds with $A(t)$ replaced by $K(t)$. In other words, we will prove that

$$\frac{A(t)-K(t)}{g(\log t)\Phi(t)} \stackrel{P}{\to}\ 0,\quad t\to\infty.$$

Since in the cases (b) and (c) $g(x)$ grows faster than $\sqrt{x}$ (see the beginning of the proof) it suffices to show that

$$\frac{A(t)-K(t)}{\sqrt{\log t}\,\Phi(t)} \stackrel{P}{\to}\ 0,\quad t\to\infty. \tag{13}$$

By Lemma 2.6 in [1],

$$\mathbb{E}\big(A(t)-K(t)\big)^2 \ \sim\ \mathtt{m}^{-1}\int_{[1,\,t]} u^{-1}\Phi(u)\mathrm{d}u,\quad t\to\infty. \tag{14}$$

Hence

$$\mathbb{E}\bigg(\frac{A(t)-K(t)}{\sqrt{\log t}\,\Phi(t)}\bigg)^2 \ \sim\ \frac{\int_{[1,\,t]}u^{-1}\Phi(u)\mathrm{d}u}{\mathtt{m}\log t\,\Phi^2(t)} \le \frac{\log t\,\Phi(t)}{\mathtt{m}\log t\,\Phi^2(t)} = \frac{1}{\mathtt{m}\Phi(t)},$$

and (13) follows by Chebyshev's inequality.
Step 3: The last step is 'depoissonization', i.e. passing from the Poisson process to the original fixed-n exponential sample. Since $K(t)$ is nondecreasing, this is easy, and the proof is omitted (see the proof of Theorem 3.5 where the depoissonization is implemented for a non-monotone function). □

Our second main result concerns 'fast' functions of slow variation Φ, which grow faster than any power of $\log t$, for instance $\Phi(t) \sim \exp(\gamma \log^{\delta} t)$ for some $\gamma > 0$ and $\delta \in (0, 1)$.

Theorem 3.3. *Suppose Condition B holds.*
(a) *Under the assumption of part (a) of Theorem 3.1*

$$\frac{K_n - \mathtt{m}^{-1} \int_{[1,\,n]} y^{-1} \Phi(y) \mathrm{d}y}{\sqrt{\mathtt{s}^2 \mathtt{m}^{-3} \log n} \Phi(n)} \stackrel{d}{\to} \int_{[0,\,\infty)} Z(y) e^{-y} \mathrm{d}y, \quad n \to \infty,$$

where $(Z(y))_{y \geq 0}$ *is the Brownian motion.*
(b) *Under the assumptions of part (b) of Theorem 3.1*

$$\frac{K_n - \mathtt{m}^{-1} \int_{[1,\,n]} y^{-1} \Phi(y) \mathrm{d}y}{\mathtt{m}^{-3/2} c(\log n) \Phi(n)} \stackrel{d}{\to} \int_{[0,\,\infty)} Z(y) e^{-y} \mathrm{d}y, \quad n \to \infty,$$

where $\big(Z(y)\big)_{y \geq 0}$ *is the Brownian motion.*
(c) *Under the assumptions of part (c) of Theorem 3.1*

$$\frac{K_n - \mathtt{m}^{-1} \int_{[1,\,n]} y^{-1} \Phi(y) \mathrm{d}y}{\mathtt{m}^{-(\alpha+1)/\alpha} c(\log n) \Phi(n)} \stackrel{d}{\to} \int_{[0,\,\infty)} Z(y) e^{-y} \mathrm{d}y, \quad n \to \infty,$$

where $\big(Z(y)\big)_{y \geq 0}$ *is the α-stable Lévy process such that $Z(1)$ has characteristic function* (5).

Remark 3.4. Set $K := \int_{[0,\,\infty)} Z(u) e^{-u} \mathrm{d}u$. By Lemma 5.1,

$$\log \mathbb{E} \exp(\mathrm{i}tK) = \int_{[0,\infty)} \log \mathbb{E} \exp(\mathrm{i}te^{-x} Z(1)) \mathrm{d}x.$$

Hence $K \stackrel{d}{=} \alpha^{-1/\alpha} Z(1)$ where the case $\alpha = 2$ corresponds to parts (a) and (b) of Theorem 3.3.

Proof. We use the same notation as in the proof of Theorem 3.1. We will only show that

$$\frac{A(e^t) - \mathtt{m}^{-1} \Big(\int_{[0,\,t]} (t - z) \mathrm{d}\varphi(z) + \varphi(0) t \Big)}{g(h(t)) \varphi(t)} \stackrel{d}{\to} K = \int_{[0,\,\infty)} Z(u) e^{-u} \mathrm{d}u, \tag{15}$$

the rest of the proof being the same as in Theorem 3.1. For any fixed $a > 0$, we have

$$\begin{aligned} \frac{A_1(t) - \mathtt{m}^{-1} \int_{[0,\,t]} (t - z) \mathrm{d}\varphi(z)}{g(h(t)) \varphi(t)} &= -\int_{[0,\,a]} W_{h(t)}(y) \mathrm{d}v_t(y) \qquad (16) \\ &- \int_{[a,\,t/h(t)]} W_{h(t)}(y) \mathrm{d}v_t(y) \\ &=: J_3(t, a) + J_4(t, a), \end{aligned}$$

where $v_t(u) := \frac{\varphi(t-yh(t))}{\varphi(t)}$. To apply Lemma 5.3 we take $X_t = W_{h(t)}$ and let Y_t and Y be random variables with $\mathbb{P}\{Y_t > u\} = v_t(u)$ and $\mathbb{P}\{Y > u\} = e^{-u}$. Then

$$J_3(t,a) := -\int_{[0,\,a]} W_{h(t)}(y)\mathrm{d}v_t(y) \overset{d}{\to} \int_{[0,\,a]} Z(y)e^{-y}\mathrm{d}y, \quad t\to\infty.$$

Hence $\lim_{a\to\infty}\lim_{t\to\infty} J_3(t,a) = K$ in distribution.

Now we intend to show that, for any $c>0$,

$$\lim_{a\to\infty}\limsup_{t\to\infty}\mathbb{P}\{|J_4(t,a)|>c\} = 0. \tag{17}$$

By Theorem 1.2 in [16], for any $\delta>0$ there exists $t_0>0$ such that

$$\frac{\mathbb{E}\big|T(t)-\mathtt{m}^{-1}t\big|}{g(t)} \le \mathbb{E}|Z(1)|+\delta$$

whenever $t\ge t_0$. Hence, for t such that $ah(t)\ge t_0$ and some $\varepsilon\in(0,1-1/\alpha)$,

$$\begin{aligned}
\mathbb{E}|J_4(t,a)| &\le \int_{[ah(t),\,\infty)} \frac{\mathbb{E}|T(y)-\mathtt{m}^{-1}y|}{g(y)}\frac{g(y)}{g(h(t))}\mathrm{d}(-v_t(y/h(t)))\\
&\le \big(\mathbb{E}|Z(1)|+\delta\big)\int_{[a,\,\infty)}\frac{g(yh(t))}{g(h(t))}\mathrm{d}(-v_t(y))\\
&\le \big(\mathbb{E}|Z(1)|+\delta\big)\,\mathrm{const}\,\mathbb{E}\eta_t^{1/\alpha+\varepsilon}1_{\{\eta_t>a\}},
\end{aligned} \tag{18}$$

where in the third line the Potter's bound (Theorem 1.5.6 in [4]) has been utilized (recall that the regular variation of g was discussed at the beginning of the proof of Theorem 3.1), and η_t is a random variable with $\mathbb{P}\{\eta_t>y\} = v_t(y)$. By Corollary 3.10.5 in [4], the auxiliary function h is unique up to the asymptotic equivalence and can be taken $h(t) = \int_{[0,\,t]}\varphi(y)\mathrm{d}y/\varphi(t)$. With such h we have

$$\mathbb{E}\eta_t = \int_{[0,\infty)} v_t(y)\mathrm{d}y = \frac{1}{h(t)}\frac{\int_{[0,t]}\varphi(y)\mathrm{d}y}{\varphi(t)} + \frac{1}{h(t)\varphi(t)}\int_{[0,1]}\frac{\Phi(y)}{y}\mathrm{d}y \;\to\; 1, \quad t\to\infty. \tag{19}$$

Note that the integral in the second term is finite in view of $\Phi'(0)<\infty$ (the latter finiteness was discussed in Step 1 of the proof of Theorem 3.1).

Now (19) implies that the family $(\eta_t^{1/\alpha+\varepsilon})_{t\ge 0}$ is uniformly integrable, and (17) follows from (18) and Markov's inequality. From this we conclude that the left-hand side of (16) converges in distribution to K. This together with (6) and (8) proves (15). $\square$

3.2 Convergence of $K_{n,1}$

We shall prove next convergence in distribution for the number of singleton blocks $K_{n,1}$. Two cases, when Condition A and Condition B holds, respectively, are treated in Theorem 3.5 and Theorem 3.7.

Theorem 3.5. *Assume that the function $t \mapsto t\Phi'(t)$ is nondecreasing and that Condition A holds with $\beta \geq 1$.*
Under the assumptions of part (a) of Theorem 3.1 we have[1]*: if $\beta > 1$ then*

$$\frac{K_{n,1} - \mathtt{m}^{-1}\Phi(n)}{n\Phi'(n)\sqrt{\mathtt{s}^2\mathtt{m}^{-3}\log n}} \overset{d}{\to} (\beta - 1)\int_{[0,1]} Z(1-y)y^{\beta-2}\mathrm{d}y, \quad n \to \infty,$$

where $(Z(y))_{y\in[0,1]}$ is the Brownian motion, and if $\beta = 1$ and $\lim_{n\to\infty} L_1(n) = \infty$ then the limiting random variable is $Z(1)$.
Under the assumptions of part (b) of Theorem 3.1 we have: if $\beta > 1$ then

$$\frac{K_{n,1} - \mathtt{m}^{-1}\Phi(n)}{\mathtt{m}^{-3/2}n\Phi'(n)c(\log n)} \overset{d}{\to} (\beta - 1)\int_{[0,1]} Z(1-y)y^{\beta-2}\mathrm{d}y, \quad n \to \infty,$$

where $(Z(y))_{y\in[0,1]}$ is the Brownian motion, and if $\beta = 1$ then the limiting random variable is $Z(1)$.
Under the assumptions of part (c) of Theorem 3.1 we have: if $\beta > 1$ then

$$\frac{K_{n,1} - \mathtt{m}^{-1}\Phi(n)}{\mathtt{m}^{-1-1/\alpha}n\Phi'(n)c(\log n)} \overset{d}{\to} (\beta - 1)\int_{[0,1]} Z(1-y)y^{\beta-2}\mathrm{d}y, \quad n \to \infty,$$

where $(Z(y))_{y\in[0,1]}$ is the α-stable Lévy process such that $Z(1)$ has characteristic function (5), and if $\beta = 1$ then the limiting random variable is $Z(1)$.

Remark 3.6. Theorem 3.5 does not cover one interesting case when $\mathtt{s}^2 < \infty$ and $\Phi(x) \sim c\log x$, as $x \to \infty$, where $c > 0$ is a constant. We conjecture that

$$\frac{K_{n,1} - \mathtt{m}^{-1}\Phi(n)}{c\log^{1/2} n} \overset{d}{\to} (\mathtt{s}^2\mathtt{m}^{-3})^{1/2}V_1 + (\mathtt{m}c)^{-1/2}V_2, \quad n \to \infty,$$

where V_1 and V_2 are independent random variables with the standard normal distribution. In combination with the proof of Theorem 3.5 this would follow once we could show that

$$\frac{K(t,1) - A^{(1)}(t)}{c\log^{1/2} t} \overset{d}{\to} (\mathtt{m}\,c)^{-1/2}V_2, \quad t \to \infty.$$

However, we have not been able to work it out.

Proof. Using (2) we have

$$A^{(1)}(e^t) = \int_{[0,\,\infty)} \varphi'(t-y)\mathrm{d}T(y) = \int_{[0,\,t]} + \int_{[t,\,\infty)} =: A_1^{(1)}(t) + A_2^{(1)}(t).$$

The function φ' is nonnegative and integrable on $(-\infty, 0]$, and the function $e^{-y}\varphi'(y)$ is nonincreasing on $\mathbb{R}$. This implies that φ' is directly Riemann integrable on $(-\infty, 0]$ (see,

[1]Suppose $\beta > 1$. According to Remark 3.2, $(\beta-1)\int_{[0,\,1]} Z(1-y)y^{\beta-2}\mathrm{d}y \overset{d}{=} (\alpha(\beta-1)+1)^{-1/\alpha}Z(1)$, where the case $\alpha = 2$ corresponds to parts (a) and (b) of Theorem 3.5.

for instance, the proof of Corollary 2.17 in [5]). Therefore, by the key renewal theorem, as $t \to \infty$,

$$\mathbb{E}A_2^{(1)}(t) \to \mathtt{m}^{-1} \int_{(-\infty, 0]} \varphi'(y)\mathrm{d}y = \Phi(1)/\mathtt{m} < \infty. \tag{20}$$

Now convergence in distribution of $A^{(1)}(t)$ with the same centering and normalization as asserted for $K_{n,1}$ (and n replaced by the continuous variable t) follows along the same lines as in the proof of Theorem 3.1 for $A(t)$.

Arguing in the same way as in the proof of Lemma 2.6 in [1] we conclude that

$$\mathbb{E}(K(t,1) - A^{(1)}(t))^2 = \mathbb{E}A^{(1)}(t).$$

Hence, according to (20) and Proposition 5.4,

$$\mathbb{E}(K(t,1) - A^{(1)}(t))^2 \ \sim \ \mathtt{m}^{-1}\Phi(t), \quad t \to \infty.$$

The function φ' is nondecreasing since $t\Phi'(t)$ was assumed such, hence by the monotone density theorem (Theorem 1.7.2 in [4]) we conclude that, as $t \to \infty$,

$$\frac{\varphi(t)}{(\varphi'(t))^2 t} \ \sim \ \frac{t^\beta L_1(t)}{\beta^2 t^{2\beta-2} L_1^2(t) t} = \frac{1}{\beta^2} \frac{1}{t^{\beta-1} L_1(t)}.$$

This converges to zero whenever $\beta > 1$ or $\beta = 1$ and $\lim_{t\to\infty} L_1(t) = \infty$. Therefore, by Chebyshev's inequality

$$\frac{K(t,1) - A^{(1)}(t)}{t\Phi'(t)\sqrt{\log t}} \ \overset{P}{\to} 0, \quad t \to \infty.$$

Since the normalization $t\Phi'(t)\sqrt{\log t}$ exhibits the slowest growth among the three normalizations arising in the theorem (see the beginning of the proof of Theorem 3.1 for explanation), under the current assumption we conclude that convergence in distribution as stated in the theorem holds with $K_{n,1}$ replaced by $K(t,1)$ and the normalizing sequences replaced by the normalizing functions.

Now we shall discuss the remaining case $\mathtt{s}^2 = \infty$, $\beta = 1$ and $\lim_{t\to\infty} L_1(t) = c \in (0, \infty)$ (note that in view of the monotonicity assumption on φ' and the relation $\varphi'(t) \sim L_1(t)$, the limit of L_1 must exist). The normalization q_n, say, claimed for $K_{n,1}$ grows not slower than $n\Phi'(n)\log^{1/2} n L_2(n)$ for some L_2 slowly varying at ∞ with $\lim_{n\to\infty} L_2(n) = \infty$. Then, Chebyshev's inequality implies

$$\frac{K(n,1) - A^{(1)}(n)}{q_n} \ \overset{P}{\to} 0, \quad n \to \infty.$$

This proves that the asserted convergence in distribution holds with $K_{n,1}$ replaced by $K(t,1)$ in this case too.

It remains to depoissonize. Let (t_k, x_k) be the atoms of a planar Poisson point process in the positive quadrant with the intensity measure given by $e^{-x}\mathrm{d}t\,\mathrm{d}x$. The process $(X_t)_{t\ge0}$ with $X_t := \sum_{t_k \le t} x_k$ is a compound Poisson process with unit intensity and jumps having the standard exponential distribution. Now, with $z \ge 0$ fixed, π_z can be identified with

the number of jumps of (X_t) occurring before time z, which implies that $(\pi_z)_{z\geq 0}$ is a homogeneous Poisson process with unit intensity. Denote by $(T_n)_{n\in\mathbb{N}}$ its arrival times. We already know that

$$\frac{K(t,1)-r(t)}{d(t)} \stackrel{d}{\to} X, \quad t\to\infty \tag{21}$$

for $r(t) := \mathtt{m}^{-1}\Phi(t)$, the case-dependent normalizing function $d(t)$ and the case-dependent random variable X. Since $K(T_n,1)=K_{n,1}$ it suffices to check that

$$\frac{K(T_n,1)-r(n)}{d(n)} \stackrel{d}{\to} X, \quad n\to\infty.$$

In the subsequent computations we will use arbitrary but fixed $x\in\mathbb{R}$. Given such x we will choose $n_0\in\mathbb{N}$ such that the sequence $(n+x\sqrt{n})_{n\geq n_0}$ is nondecreasing and every its element is not smaller than one, and the sequence $(n-x\sqrt{n})_{n\geq n_0}$ is nonnegative. Also, we will choose $t_0\in(0,\infty)$ such that $t\pm x\sqrt{t}\geq 0$ for $t\geq t_0$. With this notation all the relations that follow will be considered either for $t\geq t_0$ or $n\geq n_0$.

The function $d(t)$ is slowly varying, which implies that the convergence $\lim\limits_{t\to\infty}\frac{d(ty)}{d(t)}=1$ holds locally uniformly in y. In particular,

$$\lim_{t\to\infty}\frac{d(t\pm x\sqrt{t})}{d(t)}=1. \tag{22}$$

The function $r(t)$ has the following property

$$\lim_{t\to\infty}\frac{r(t\pm x\sqrt{t})-r(t)}{d(t)}=0. \tag{23}$$

Indeed, the function $t\mapsto\Phi'(t)$ is nonincreasing, and using the mean value theorem we conclude that

$$\frac{r(t+x\sqrt{t})-r(t)}{d(t)}\leq\frac{\mathtt{m}^{-1}x\sqrt{t}\Phi'(t)}{t\Phi'(t)}o(1),\quad \frac{r(t)-r(t-x\sqrt{t})}{d(t)}\leq\frac{\mathtt{m}^{-1}x\sqrt{t}\Phi'(t-x\sqrt{t})}{t\Phi'(t)}o(1).$$

By the monotone density theorem (Theorem 1.7.2 in [4]), the function $\Phi'(t)$ is regularly varying at ∞ with index -1. Hence $\lim\limits_{t\to\infty}\frac{\Phi'(t-x\sqrt{t})}{\Phi'(t)}=1$, and the right-hand side of the last inequality tends to zero, as $t\to\infty$.

Now (22) and (23) ensure that (21) is equivalent to

$$\frac{K(t\pm x\sqrt{t},1)-r(t)}{d(t)} \stackrel{d}{\to} X, \quad t\to\infty. \tag{24}$$

We will need the following observation

$$\frac{K(t+x\sqrt{t})-K(t-x\sqrt{t})}{d(t)} \stackrel{P}{\to} 0, \quad t\to\infty, \tag{25}$$

which can be proved as follows. Since $K(t)$ is nondecreasing it suffices to show that the expectation of the left-hand side converges to zero. To this end, write

$$\begin{aligned}
&\mathbb{E}\Big(K(t+x\sqrt{t})-K(t-x\sqrt{t})\Big)\\
=\;&\mathbb{E}\int_{[0,\infty)}\Big(\varphi\big(\log(t+x\sqrt{t})-y\big)-\varphi\big(\log(t-x\sqrt{t})-y\big)\Big)\mathrm{d}T(y)\\
=\;&\mathbb{E}\int_{[0,\,\log(t+x\sqrt{t})]}\Big(\varphi\big(\log(t+x\sqrt{t})-y\big)-\varphi\big(\log(t-x\sqrt{t})-y\big)\Big)\mathrm{d}T(y)+O(1)\\
\le\;&\log\left(\frac{t+x\sqrt{t}}{t-x\sqrt{t}}\right)\mathbb{E}\int_{[0,\,\log(t+x\sqrt{t})]}\varphi'\big(\log(t+x\sqrt{t})-y\big)\mathrm{d}T(y)+O(1)\\
\sim\;&\frac{2x}{\sqrt{t}}\,\mathbb{E}\int_{[0,\,\log(t+x\sqrt{t})]}\varphi'\big(\log(t+x\sqrt{t})-y\big)\mathrm{d}T(y)\\
\sim\;&\frac{2x}{\mathtt{m}\sqrt{t}}\int_{[0,\,\log(t+x\sqrt{t})]}\varphi'(y)\mathrm{d}y\;\sim\;\frac{2x}{\mathtt{m}\sqrt{t}}\Phi(t+x\sqrt{t})\sim\frac{2x}{\mathtt{m}\sqrt{t}}\Phi(t),\quad t\to\infty.
\end{aligned}$$

Here the third line is a consequence of the key renewal theorem (see the paragraph preceding formula (20) for more details). The fourth line follows from the mean value theorem and the monotonicity of φ'. While Proposition 5.4 justifies the first asymptotic equivalence in the sixth line of the last display, the last equivalence in that line is implied by the slow variation of Φ (see the sentence preceding (22) for the explanation). Now (25) follows from the last asymptotic relation and the observation $\lim_{t\to\infty}\frac{t\Phi'(t)}{\Phi(t)}\sqrt{t}=\infty$, the latter being trivial as the first factor is slowly varying.

Set $D_n(x):=\{|T_n-n|>x\sqrt{n}\}$. Since $K(t)$ and $L(t):=K(t)-K(t,1)$ are nondecreasing, we have, for any $\varepsilon>0$,

$$\begin{aligned}
\mathbb{P}\Big\{\frac{K(T_n,1)-K(n-x\sqrt{n},1)}{d(n)}>2\varepsilon\Big\}\;&=\;\mathbb{P}\Big\{\frac{K(T_n)-L(T_n)-K(n-x\sqrt{n},1)}{d(n)}>2\varepsilon\Big\}\\
&=\;\mathbb{P}\big\{\ldots 1_{D_n^c(x)}+\ldots 1_{D_n(x)}>2\varepsilon\big\}\\
&\le\;\mathbb{P}\Big\{\frac{K(n+x\sqrt{n})-L(n-x\sqrt{n})-K(n-x\sqrt{n},1)}{d(n)}>\varepsilon\Big\}\\
&+\;\mathbb{P}\big\{\ldots 1_{D_n(x)}>\varepsilon\big\}\\
&\le\;\mathbb{P}\Big\{\frac{K(n+x\sqrt{n})-K(n-x\sqrt{n})}{d(n)}>\varepsilon\Big\}+\mathbb{P}\big(D_n(x)\big).
\end{aligned}$$

Hence, by (25) and the central limit theorem

$$\limsup_{n\to\infty}\mathbb{P}\Big\{\frac{K(T_n,1)-K(n-x\sqrt{n},1)}{a(n)}>2\varepsilon\Big\}\le\mathbb{P}\{|\mathcal{N}(0,1)|>x\},\tag{26}$$

where $\mathcal{N}(0,1)$ denotes a random variable with the standard normal distribution. Since

the law of X is continuous, we conclude that, for any $y \in \mathbb{R}$ and any $\varepsilon > 0$,

$$\begin{aligned}
\limsup_{n\to\infty} \mathbb{P}\Big\{\frac{K(T_n,1)-r(n)}{d(n)} > y\Big\} &\leq \limsup_{n\to\infty} \mathbb{P}\Big\{\frac{K(T_n,1)-K(n-x\sqrt{n},1)}{d(n)} > 2\varepsilon\Big\} \\
&+ \lim_{n\to\infty} \mathbb{P}\Big\{\frac{K(n-x\sqrt{n},1)-r(n)}{d(n)} > y-2\varepsilon\Big\} \\
&\overset{(24),(26)}{\leq} \mathbb{P}\big\{|\mathcal{N}(0,1)| > x\big\} + \mathbb{P}\big\{X > y-2\varepsilon\big\}.
\end{aligned}$$

Letting now $x \to \infty$ and then $\varepsilon \downarrow 0$ gives

$$\limsup_{n\to\infty} \mathbb{P}\Big\{\frac{K(T_n,1)-r(n)}{d(n)} > y\Big\} \leq \mathbb{P}\big\{X > y\big\}.$$

Arguing similarly we infer

$$\limsup_{n\to\infty} \mathbb{P}\Big\{\frac{K(n+x\sqrt{n},1)-K(T_n,1)}{d(n)} > 2\varepsilon\Big\} \leq \mathbb{P}\big\{|\mathcal{N}(0,1)| > x\big\} \tag{27}$$

and then

$$\begin{aligned}
\liminf_{n\to\infty} \mathbb{P}\Big\{\frac{K(T_n,1)-r(n)}{d(n)} > y\Big\} &\geq \lim_{n\to\infty} \mathbb{P}\Big\{\frac{K(n+x\sqrt{n},1)-r(n)}{d(n)} > y+2\varepsilon\Big\} \\
&- \limsup_{n\to\infty} \mathbb{P}\Big\{\frac{K(n+x\sqrt{n},1)-K(T_n,1)}{d(n)} > 2\varepsilon\Big\} \\
&\overset{(24),(27)}{\geq} \mathbb{P}\big\{X > y+2\varepsilon\big\} - \mathbb{P}\big\{|\mathcal{N}(0,1)| > x\big\}.
\end{aligned}$$

Letting $x \to \infty$ and then $\varepsilon \downarrow 0$ we arrive at

$$\liminf_{n\to\infty} \mathbb{P}\Big\{\frac{K(T_n,1)-r(n)}{d(n)} > y\Big\} \geq \mathbb{P}\big\{X > y\big\}.$$

The proof is complete. $\square$

Theorem 3.7. *Assume that the function $t \mapsto t\Phi'(t)$ is nondecreasing and that Condition B holds.*
Under the assumptions of part (a) of Theorem 3.3[2]

$$\frac{K_{n,1} - \mathtt{m}^{-1}\Phi(n)}{n\Phi'(n)\sqrt{\mathtt{s}^2\mathtt{m}^{-3}h(\log n)}} \overset{d}{\to} \int_{[0,\infty)} Z(y)e^{-y}\mathrm{d}y, \quad n \to \infty,$$

where $(Z(y))_{y\geq 0}$ is the Brownian motion.
Under the assumptions of part (b) of Theorem 3.3

$$\frac{K_{n,1} - \mathtt{m}^{-1}\Phi(n)}{\mathtt{m}^{-3/2}n\Phi'(n)c(h(\log n))} \overset{d}{\to} \int_{[0,\infty)} Z(y)e^{-y}\mathrm{d}y, \quad n \to \infty,$$

[2]See Remark 3.4 for the identification of the laws of $\int_{[0,\infty)} Z(y)e^{-y}\mathrm{d}y$.

where $(Z(y))_{y\geq 0}$ is the Brownian motion.
Under the assumptions of part (c) of Theorem 3.3

$$\frac{K_{n,1}-\mathtt{m}^{-1}\Phi(n)}{\mathtt{m}^{-1-1/\alpha}n\Phi'(n)c(h(\log n))} \ \overset{d}{\to}\ \int_{[0,\infty)} Z(y)e^{-y}\mathrm{d}y,\quad n\to\infty,$$

where $(Z(y))_{y\geq 0}$ is the α-stable Lévy process such that $Z(1)$ has characteristic function (5).

Proof. By Theorem 3.10.11 in [4] φ' belongs to de Haan's class Γ. For later use, note that this implies

$$\lim_{t\to\infty}\varphi'(t)=\infty. \tag{28}$$

By Corollary 3.10.7 in [4] one can take h as the auxiliary function of φ'. With this at hand the proof of convergence in distribution of $A^{(1)}(t)$ with the same centering and normalization as claimed for $K_{n,1}$ (but with discrete argument n replaced by continuous argument t) literally repeats the proof of Theorem 3.3, thus omitted.

The next step is to prove that

$$\frac{K(t,1)-A^{(1)}(t)}{d(t)} \ \overset{P}{\to} 0,\quad t\to\infty,$$

where, depending on the context, $d(t)$ equals either

$$\text{const}\, t\Phi'(t)\sqrt{h(\log t)}\quad \text{or}\quad \text{const}\, t\Phi'(t)c(h(\log t)).$$

Since the function $t\Phi'(t)\sqrt{h(\log t)}$ grows slower than the other one it suffices to prove that

$$\frac{K(t,1)-A^{(1)}(t)}{t\Phi'(t)\sqrt{h(\log t)}} \ \overset{P}{\to} 0,\quad t\to\infty. \tag{29}$$

From the proof of Theorem 3.5 we know that

$$\mathbb{E}(K(t,1)-A^{(1)}(t))^2\ \sim\ \mathtt{m}^{-1}\Phi(t),\quad t\to\infty.$$

By Corollary 3.10.5 in [4],

$$h(t)\sim\frac{\varphi(t)}{\varphi'(t)},\quad t\to\infty.$$

Therefore, using (28) at the last step,

$$\frac{\varphi(t)}{(\varphi'(t))^2h(t)}\ \sim\ \frac{1}{\varphi'(t)}\ \to\ 0,\quad t\to\infty,$$

and relation (29) follows by Chebyshev's inequality.

By Lemma 5.2, the functions $d(t)$ are slowly varying at ∞. Keeping this in mind, the depoissonization step runs exactly the same route as in the proof of Theorem 3.5. $\square$

4 The compound Poisson case

In this section we assume that S is a compound Poisson process whose Lévy measure ν is a probability measure. This does not reduce generality, since the range $\mathcal{R}$ is not affected by the normalization of ν. Let $-\log W_1,\ \ -\log W_2, \dots$ (where $0 < W_j < 1$ a.s.) be the sizes of the consecutive jumps of S, which are independent random variables with distribution ν. Define a zero-delayed random walk $(R_k)_{k\geq 0}$ with such increments $-\log W_k$. In these terms, the Laplace exponent of S is $\widehat{\Phi}(t) = 1 - \mathbb{E}e^{-t(1-W_1)}$.

The argument exploited in Section 3 extends smoothly when the variance of $S(1)$ is infinite. Otherwise the problem arises that the terminal value $A(n)$ of the compensator does not absorb enough of the variability of K_n. The continuous-time compensator process carries extra variability coming from the exponential waiting times between the jumps of S. Without going into details we only mention that the excessive variability is seen from the asymptotics

$$\mathbb{E}\big(K(t) - A(t)\big)^2 \ \sim\ \mathtt{m}^{-1}\log t, \quad t \to \infty, \tag{30}$$

where $\mathtt{m} = \mathbb{E}S(1) = \mathbb{E}|\log W_1|$.

To circumvent the complication we note that in the case of finite Lévy measure the setting is intrinsically discrete-time, hence it is natural to replace $T(y)$ in (1) by

$$\rho(y) := \inf\{k \in \mathbb{N}_0 : R_k > y\}$$

and to consider a *discrete-time compensator*. Denote by C_k the event that the interval $[R_{k-1}, R_k]$ is occupied by at least one point of the Poisson process $(\pi_t(u))_{u\geq 0}$. Then $K(t) = \sum_{k\geq 1} 1_{C_k}$, and we define the discrete-time compensator by

$$B(t) := \sum_{k\geq 1} \mathbb{P}\{C_k | R_{k-1}\} = \sum_{k\geq 1} \widehat{\Phi}\big(te^{-R_{k-1}}\big) = \int_{[0,\infty)} \widehat{\Phi}\big(te^{-y}\big)\mathrm{d}\rho(y).$$

Indeed, $\mathbb{P}\{C_k|R_{k-1}, W_k\} = 1 - \exp\big(-te^{-R_{k-1}}(1 - W_k)\big)$ which entails $\mathbb{P}\{C_k|R_{k-1}\} = \widehat{\Phi}(te^{-R_{k-1}})$, thus justifying the second equality above. For the discrete-time compensator we have

$$\mathbb{E}(K(t) - B(t))^2 = \int_{[0,\infty)} \widehat{\Phi}(te^{-y})(1 - \widehat{\Phi}(te^{-y}))\mathrm{d}\mathbb{E}\rho(y) = o(\log t), \quad t \to \infty,$$

which compared with (30) shows that $B(t)$ approximates $K(t)$ better than $A(t)$. Furthermore, by the key renewal theorem the integral converges to

$$\mathtt{m}^{-1} \int_{[0,\infty)} \widehat{\Phi}(y)\big(1 - \widehat{\Phi}(y)\big)y^{-1}\mathrm{d}y$$

provided that $\int_{[1,\infty)} \big(1 - \widehat{\Phi}(y)\big)y^{-1}\mathrm{d}y < \infty$, and by Proposition 5.4 it is asymptotic to $\mathtt{m}^{-1}\int_{[0,\log t]} \big(1 - \widehat{\Phi}(e^y)\big)\mathrm{d}y$ otherwise. These findings allow us to simplify the proof of the following result obtained previously in [8].

Theorem 4.1. (a) *If* $\sigma^2 = \mathrm{Var}(\log W) < \infty$ *then for*

$$b_n = \frac{1}{\mathtt{m}} \int_1^n \frac{\widehat{\Phi}(z)}{z} \mathrm{d}z \quad or \quad b_n = \frac{1}{\mathtt{m}} \int_0^{\log n} \mathbb{P}\{|\log(1-W)| \leq z\} \mathrm{d}z, \tag{31}$$

where $\mathtt{m} = \mathbb{E}S(1) = \mathbb{E}|\log W|$, *and for*

$$a_n = \sqrt{\frac{\sigma^2}{\mathtt{m}^3} \log n},$$

the limiting distribution of $\frac{K_n - b_n}{a_n}$ *is standard normal.*
(b) *If* $\sigma^2 = \infty$ *and*

$$\int_0^x y^2 \nu(\mathrm{d}y) \ \sim \ L(x), \quad x \to \infty,$$

for some L *slowly varying at* ∞, *then, with* b_n *given in* (31) *and*

$$a_n = \mathtt{m}^{-3/2} c(\log n),$$

where $c(x)$ *is any positive function satisfying* $\lim_{x\to\infty} xL(c(x))/c^2(x) = 1$, *the limiting distribution of* $\frac{K_n - b_n}{a_n}$ *is standard normal.*
(c) *If* ν *satisfies* (4) *then, with* b_n *as in* (31) *and*

$$a_n = \mathtt{m}^{-1-1/\alpha} c(\log n), \tag{32}$$

where $c(x)$ *is any positive function satisfying* $\lim_{x\to\infty} xL(c(x))/c^\alpha(x) = 1$, *the limiting distribution of* $(K_n - b_n)/a_n$ *is the* α*-stable law with characteristic function* (5).

Proof. Let g and Z be as defined at the beginning of the proof of Theorem 3.1. We only give a proof of the poissonized version of the result, with K_n replaced by $B(t)$. Recalling the notation $\varphi(y) = \Phi(e^y)$ and noting that φ is integrable in the neighborhood of $-\infty$, an appeal to Theorem 4.1 in [15] gives

$$\frac{B(e^t) - \mathtt{m}^{-1} \int_{[1,\, e^t]} \big(\Phi(y)/y\big) \mathrm{d}y}{g(t)} = \frac{\int_{[0,\,\infty)} \varphi(t-y) \mathrm{d}\rho(y) - \mathtt{m}^{-1} \int_{[0,\,t]} \varphi(y) \mathrm{d}y}{g(t)} \ \overset{d}{\to} \ Z(1), \ \ t \to \infty.$$

Lemma 5.5 with $V = 1 - W_1$ ensures that the centering

$$\mathtt{m}^{-1} \int_{[0,\,t]} \varphi(u) \mathrm{d}u = \mathtt{m}^{-1} \int_{[0,\,t]} \big(1 - \mathbb{E}\exp(-e^u(1-W_1))\big) \mathrm{d}u$$

can be safely replaced by

$$\mathtt{m}^{-1} \int_{[0,\,t]} \mathbb{P}\{|\log(1-W)| \leq u\} \mathrm{d}u = \mathtt{m}^{-1} \int_{[0,\,t]} \mathbb{P}\{1 - W \geq e^{-u}\} \mathrm{d}u,$$

because the absolute value of their difference is $O(1)$ and $\lim_{t\to\infty} g(t) = \infty$. Replacing e^t by t completes the proof. $\square$

5 Appendix

The first auxiliary result concerns the laws of some Riemann integrals of the Lévy processes.

Lemma 5.1. *Let q be a Riemann integrable function on $[0,1]$ and $(Z(y))_{y\in[0,1]}$ a Lévy process with $g(t) := \log \mathbb{E} \exp(\mathrm{i}tZ(1))$. Then*

$$\mathbb{E} \exp\left(\mathrm{i}t \int_{[0,1]} q(y)Z(y)\mathrm{d}y\right) = \exp\left(\int_{[0,1]} g\left(t\int_{[y,1]} q(z)\mathrm{d}z\right)\mathrm{d}y\right), \quad t \in \mathbb{R}. \tag{33}$$

Similarly, for q a directly Riemann integrable function on $[0,\infty)$ and $(Z(y))_{y\geq 0}$ a Lévy process it holds that

$$\mathbb{E} \exp\left(\mathrm{i}t \int_{[0,\infty)} q(y)Z(y)\mathrm{d}y\right) = \exp\left(\int_{[0,\infty)} g\left(t\int_{[y,\infty)} q(z)\mathrm{d}z\right)\mathrm{d}y\right), \quad t \in \mathbb{R}.$$

Proof. We only prove the first assertion. The integral in the left-hand side of (33) exists as a Riemann integral and as such can be approximated by Riemann sums

$$\begin{aligned} n^{-1}\sum_{k=1}^{n} q(k/n)Z(k/n) &= \sum_{k=1}^{n}\Big(Z(k/n) - Z((k-1)/n)\Big)\Big(n^{-1}\sum_{j=k}^{n} q(j/n)\Big) \\ &=: \sum_{k=1}^{n}\Big(Z(k/n) - Z((k-1)/n)\Big)a_{k,n} =: I_n \end{aligned}$$

Since Z has independent and stationary increments, we conclude that

$$\log \mathbb{E} \exp(\mathrm{i}tI_n) = n^{-1}\sum_{k=1}^{n} g(ta_{k,n}).$$

Letting $n \to \infty$ we arrive at (33), by Lévy's continuity theorem for characteristic functions. □

Lemma 5.2 collects some useful properties of the functions Φ satisfying Condition B.

Lemma 5.2. *Suppose Condition B holds. Then the functions $\Phi(t)$ and $h(\log t)$ are slowly varying at ∞. The function $t \mapsto t\Phi'(t)$ is slowly varying at ∞, whenever it is nondecreasing.*

Proof. By Proposition 3.10.6 and Theorem 2.11.3 in [4], the function $h(\log t)$ is slowly varying. As was already mentioned in the proof of Theorem 3.3, without loss of generality the auxiliary function h can be taken $h(t) = \int_{[0,t]} \varphi(y)\mathrm{d}y/\varphi(t)$. By the representation theorem for slowly varying functions (Theorem 1.3.1 in [4]), the function $t \mapsto \int_{[0,\log t]} \varphi(y)\mathrm{d}y$ is slowly varying. Hence $\Phi(t) = \varphi(\log t)$ is slowly varying as well.

By Theorem 3.10.11 and Corollary 3.10.7 in [4], φ' belongs to de Haan's class Γ with the auxiliary function h_1 such that $h_1(t) \sim h(t)$, $t \to \infty$. By Corollary 3.10.5 in [4], $t\Phi'(t) \sim \Phi(t)/h(\log t)$, $t \to \infty$. Since both numerator and denominator are slowly varying functions, the function $t \mapsto t\Phi'(t)$ is slowly varying. □

The following lemma was a basic ingredient in the proof of our main results (Theorem 3.1 and the like).

Lemma 5.3. *Assume that $X_t(\cdot) \Rightarrow X(\cdot)$, as $t \to \infty$, in $D[0,\infty)$ in the Skorohod M_1 or J_1 topology. Assume also that, as $t \to \infty$, $Y_t \xrightarrow{d} Y$, where (Y_t) is a family of nonnegative random variables such that $\mathbb{P}\{Y_t = 0\}$ may be positive, and Y has an absolutely continuous distribution. Then, for $a > 0$,*

$$\int_{[0,\,a]} X_t(u)\mathbb{P}\{Y_t \in \mathrm{d}u\} \ \xrightarrow{d} \ \int_{[0,\,a]} X(u)\mathbb{P}\{Y \in \mathrm{d}u\}, \quad t \to \infty.$$

Proof. It suffices to prove that

$$\lim_{t\to\infty} \mathbb{E}h_t(Y_t) = \mathbb{E}h(Y) \tag{34}$$

whenever $h_t \to h$ in $D[0,\infty)$ in the M_1 or J_1 topology, for the desired result then follows by the continuous mapping theorem.
Since h restricted to $[c,d]$ is in $D[c,d]$ the set $\mathcal{D}_h$ of its discontinuities is at most countable. By Lemma 12.5.1 in [19], convergence in the M_1 topology (hence in the J_1 topology) implies the local uniform convergence at all continuity points of the limit distribution. Hence

$$E := \{y : \text{there exists } y_t \text{ such that } \lim_{t\to\infty} y_t = y, \text{but } \lim_{n\to\infty} h_t(y_t) \neq h(y)\} \subseteq \mathcal{D}_h,$$

and we conclude that $\mathbb{P}\{Y \in E\} = 0$. Now (34) follows by Theorem 5.5 in [2]. $\square$

Proposition 5.4 is a slight extension of Theorem 4 in [18] and Lemma 5.4 in [14].

Proposition 5.4. *Let v be a nonnegative function with $\lim_{t\to\infty} \int_{[0,\,t]} v(z)\mathrm{d}z = \infty$. Assume further that v is either nondecreasing with*

$$\lim_{t\to\infty} \frac{v(t)}{\int_{[0,\,t]} v(z)\mathrm{d}z} = 0,$$

or nonincreasing. If $\mathtt{m} = \mathbb{E}S(1) < \infty$ then

$$\int_{[0,\,t]} v(t-z)\mathrm{d}\mathbb{E}T(z) \ \sim \ \mathtt{m}^{-1} \int_{[0,\,t]} v(z)\mathrm{d}z, \quad t \to \infty, \tag{35}$$

provided the subordinator S is nonarithmetic. The asymptotic relation holds with additional factor δ for S arithmetic subordinator with span $\delta > 0$.

Sgibnev [18] and Iksanov [14] assumed that $T(u)$ is the first passage time through the level u by a random walk with nonnegative steps. The transition to the present setting is easy in the view of $\mathbb{E}T(u) = \mathbb{E}N^*(u) + \delta_0(u)$, where $N^*(u)$ is the first passage time through the level u by a zero-delayed random walk with the generic increment ξ having the distribution

$$\mathbb{P}\{\xi \in \mathrm{d}x\} = \int_{[0,\infty)} \mathbb{P}\{S(t) \in \mathrm{d}x\}e^{-t}\mathrm{d}t.$$

It is clear that if the law of $S(1)$ is arithmetic with span $\delta > 0$ (respectively, nonarithmetic) then the same is true for the law of ξ.

The next lemma was used in the proof of Theorem 4.1.

Lemma 5.5. *For $x > 0$ and a random variable $V \in (0, 1)$,*

$$-\int_{[0,\,1]} \frac{1-e^{-y}}{y}\mathrm{d}y \le f_1(x) - f_2(x) \le \int_{[1,\,\infty)} \frac{e^{-y}}{y}\mathrm{d}y,$$

where

$$f_1(x) := \int_{[0,\,x]} \mathbb{E}\exp(-e^y V)\mathrm{d}y \quad \textit{and} \quad f_2(x) := \int_{[0,\,x]} \mathbb{P}\{V < e^{-y}\}\mathrm{d}y.$$

Proof. For fixed $z > 0$ define $r(x) = x \wedge z$, $x \in \mathbb{R}$. This function is subadditive on $[0, \infty)$ and nondecreasing. Hence, for $x \ge 0$ and $y \in \mathbb{R}$ we have

$$r((x+y)^+) \le r(x+y^+) \le r(x) + r(y^+) \le r(x) + y^+$$

and

$$\begin{aligned} r((x+y)^+) - r(x) &\ge r(x-y^-) - r(x) \\ &= (r(x-y^-) - r(x))1_{\{x \le z\}} + (r(x-y^-) - r(x))1_{\{x>z,\, x-y^- \le z\}} \\ &= -y^- 1_{\{x\le z\}} + (x - y^- - z)1_{\{x>z,\, x-y^-\le z\}} \\ &\ge -y^- 1_{\{x \le z\}} - y^- 1_{\{x>z,\, x-y^- \le z\}} \\ &\ge -y^-. \end{aligned}$$

Thus we have proved that, for $x \ge 0$ and $y \in \mathbb{R}$

$$-y^- \le r((x+y)^+) - r(x) \le y^+. \tag{36}$$

Since $f_2(z) = \mathbb{E}(|\log V| \wedge z)$ and

$$f_1(z) = \int_{[0,\,z]} \mathbb{P}\{E/V > e^y\}\mathrm{d}y = \int_{[0,\,z]} \mathbb{P}\{|\log V| + \log E > y\}\mathrm{d}y = \mathbb{E}((|\log V| + \log E)^+ \wedge z),$$

where E is a random variable with the standard exponential distribution which is independent of V, (36) entails

$$-\mathbb{E}\log^- E \le f_1(z) - f_2(z) \le \mathbb{E}\log^+ E.$$

The proof is complete. $\square$

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